The Transcendence of e and π

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Algebraic vs Transcendental

Recall that a complex number $\alpha \in \mathbb{C}$ is called algebraic if there exists a nonzero polynomial $f \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$. The degree of α is defined to be the degree of the extension $\mathbb{Q}(\alpha)/\mathbb{Q}$. It is easy to find algebraic numbers of any degrees because of the following theorem.

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Examples

$\sqrt{2} + i$, $\sqrt{\sqrt[3]{2} + \sqrt[5]{3}}$, $\sqrt[3]{10}/\sqrt[5]{12}$, $\sqrt{3}e^{2\pi i/7} - 2e^{2\pi i/5}$ are all algebraic.

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A complex number $\alpha \in \mathbb{C}$ is called transcendental if $\alpha \notin \overline{\mathbb{Q}}$. Unlike $\overline{\mathbb{Q}}$, the set of all transcendental numbers in $\mathbb{C}$ does not possess good algebraic structures.

Basic Questions

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Question 3

Is $e = 2.718...$ transcendental? What about $\pi = 3.141...$?

History

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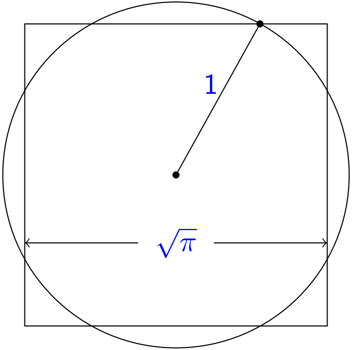
- Cantor (1874) showed that almost all complex numbers are transcendental.
- Lindemann (1882) established the transcendence of π.

Squaring the Circle

A geometric problem proposed by ancient Greeks asks whether it is possible to construct a square with the same area as a given circle with compass and straightedge only. Algebraically, one is asked to find an algebraic number x satisfying the equation $x^2 = \pi$. Lindemann's result implies that $\sqrt{\pi}$ is transcendental, proving that this problem is unsolvable.



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Preliminary Lemmas

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Let R be a domain and $c \in R$. Suppose $f \in R[x]$. Then $f^{(j)}(c)/j! \in R$ for all $j \geq 0$.

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Proof.

Since $R[x] = R[x-c]$, we may write

$$f(x) = \sum_{j=0}^{m} a_j (x-c)^j,$$

where $m = \deg f$ and $a_0, \ldots, a_m \in R$. It follows that $a_j = f^{(j)}(c)/j! \in R$ for all $0 \leq j \leq m$. For $j > m$ we have $f^{(j)}(c) = 0$. $\square$

Preliminary Lemmas

Let $f(x) = \sum_{r=0}^{m} a_r x^r$ be a polynomial of degree m with complex coefficients, and let $\bar{f}(x) := \sum_{r=0}^{m} |a_r| x^r$. For $z \in \mathbb{C}$, define

$$I(z) := \int_0^z e^{z-t} f(t)\, dt.$$

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$$I(z) := \int_0^z e^{z-t} f(t)\, dt.$$

Lemma 2

$I(z)$ satisfies the following properties:

(i) $I(z) = e^z \sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m} f^{(j)}(z)$.

(ii) $|I(z)| \leq |z| e^{|z|} \bar{f}(|z|)$.

Transcendence of e

Theorem 1 (Hermite, 1873)

e is transcendental.

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Proof.

Assume $e \in \overline{\mathbb{Q}}$. Then there exist $a_0, \ldots, a_n \in \mathbb{Z}$ such that $a_0 \neq 0$ and

$$\sum_{k=0}^{n} a_k e^k = 0. \tag{1}$$

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$$\sum_{k=0}^{n} a_k e^k = 0. \tag{1}$$

Define an auxiliary polynomial

$$f(x) := x^{p-1}(x-1)^p \cdots (x-n)^p,$$

where p is a prime, and consider

$$J := \sum_{k=0}^{n} a_k I(k).$$

Proof (Cont).

It is clear that $m := \deg f = (n+1)p - 1$. By Lemma 2 and (1) we have

$$J = \sum_{k=0}^{n} a_k e^k \sum_{j=0}^{m} f^{(j)}(0) - \sum_{k=0}^{n} a_k \sum_{j=0}^{m} f^{(j)}(k) = -\sum_{j=0}^{m} \sum_{k=0}^{n} a_k f^{(j)}(k).$$

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Observe that $f^{(j)}(k) = 0$ in each of the following two cases:

- $0 \leq j < p$ and $1 \leq k \leq n$.
- $0 \leq j < p - 1$ and $k = 0$.

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Observe that $f^{(j)}(k) = 0$ in each of the following two cases:

- $0 \le j < p$ and $1 \le k \le n$.
- $0 \le j < p-1$ and $k = 0$.

Note also that

$$f^{(p-1)}(0) = (p-1)!(-1)^{np}(n!)^p.$$

By Lemma 1, $f^{(j)}(k) \equiv 0 \pmod{j!}$ for all $j \ge p$ and $0 \le k \le n$. Hence, we have

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Proof (Cont).

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$$\bar{f}(k) \leq k^{p-1}(k+1)^p ... (k+n)^p \leq ((2n)!)^p$$

for all $0 \leq k \leq n$.

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$$\bar{f}(k) \leq k^{p-1}(k+1)^p ... (k+n)^p \leq ((2n)!)^p$$

for all $0 \leq k \leq n$. It follows from Lemma 2 that

$$|J| \leq \sum_{k=0}^{n} |a_k||I(k)| \leq \sum_{k=0}^{n} k|a_k|e^k \bar{f}(k) \ll ((2n)!)^p.$$

So $(p-1)! \ll ((2n)!)^p$. This is false when p is sufficiently large. $\square$

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Algebraic Numbers and Algebraic Integers

We need a few more concepts and facts about algebraic numbers.

1. The Galois conjugates of an algebraic number α are the complex zeros of the minimal polynomial of α (over $\mathbb{Q}$).
2. A complex number α is called an algebraic integer if it is a zero of a monic non-constant polynomial $f \in \mathbb{Z}[x]$. One can show that α is an algebraic integer if and only if its minimal polynomial has integer coefficients.
3. The set $\overline{\mathbb{Z}} \subseteq \overline{\mathbb{Q}}$ of all algebraic integers in $\mathbb{C}$ is a domain with the property that $\overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}$.
4. Suppose that $\alpha \in \overline{\mathbb{Q}}$ is a zero of $f \in \mathbb{Z}[x]$ with leading coefficient $c \neq 0$. Then $c\alpha \in \overline{\mathbb{Z}}$.

Symmetric Polynomials

Let $n \in \mathbb{N}_+$. For each $1 \leq k \leq n$, the k^{th} elementary symmetric polynomial in n variables is defined by

$$e_k(x_1, ..., x_n) := \sum_{1 \leq i_1 < ... < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

A polynomial $P(x_1, ..., x_n)$ is called a symmetric polynomial if $P(x_{\sigma(1)}, ..., x_{\sigma(n)}) = P(x_1, ..., x_n)$ for all permutation $\sigma \in S_n$.

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Theorem (Fundamental Theorem of Symmetric Polynomials)

Let R be a commutative ring. Then for every symmetric polynomial $P \in R[x_1, ..., x_n]$, there exists a unique $Q \in R[x_1, ..., x_n]$ such that

$$P(x_1, ..., x_n) = Q(e_1(x_1, ..., x_n), ..., e_n(x_1, ..., x_n)).$$

Symmetric Polynomials

Lemma 3

Let $f, g \in \mathbb{Z}[x]$ *with* g *monic. Suppose that* $\alpha_1, \ldots, \alpha_n$ *are the zeros of* g*. Then for every* $j \geq 0$,

$$\frac{1}{j!} \sum_{k=1}^{n} f^{(j)}(\alpha_k) \in \mathbb{Z}.$$

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$$\frac{1}{j!}\sum_{k=1}^{n} f^{(j)}(\alpha_k) \in \mathbb{Z}.$$

Proof.

Consider the symmetric polynomial

$$P(x_1, \ldots, x_n) := \sum_{k=1}^{n} f^{(j)}(x_k) \in \mathbb{Z}[x_1, \ldots, x_n].$$

There exists $Q \in \mathbb{Z}[x_1, \ldots, x_n]$ such that

$$P(x_1, \ldots, x_n) = Q(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)).$$

Symmetric Polynomials

Proof (Cont).

Let $g(x) = x^n + a_{n-1}x^{n-1} + ... + a_0 \in \mathbb{Z}[x]$. Then for $1 \leq r \leq n$, we have $e_r(\alpha_1, ..., \alpha_n) = (-1)^r a_{n-r} \in \mathbb{Z}$. Hence

$$P(\alpha_1, ..., \alpha_n) = Q(e_1(\alpha_1, ..., \alpha_n), ..., e_n(\alpha_1, ..., \alpha_n)) \in \mathbb{Z}.$$

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$$P(\alpha_1, ..., \alpha_n) = Q(e_1(\alpha_1, ..., \alpha_n), ..., e_n(\alpha_1, ..., \alpha_n)) \in \mathbb{Z}.$$

Since $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Z}}$, it follows from Lemma 1 that

$$\frac{1}{j!}P(\alpha_1, ..., \alpha_n) = \sum_{k=1}^{n} \frac{f^{(j)}(\alpha_k)}{j!} \in \overline{\mathbb{Z}}.$$

Therefore, we have

$$\frac{1}{j!}P(\alpha_1, ..., \alpha_n) \in \overline{\mathbb{Z}} \cap \mathbb{Q} = \mathbb{Z}.$$

This completes the proof. $\square$

Transcendence of π

Theorem 2 (Lindemann, 1882)

π is transcendental.

Proof.

Assume $\pi \in \overline{\mathbb{Q}}$. Then $\theta := \pi i \in \overline{\mathbb{Q}}$. Let $\theta_1 = \theta, \theta_2, ..., \theta_d$ be the Galois conjugates of θ.

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$$(e^{\theta_1} + 1) \cdots (e^{\theta_d} + 1) = 0.$$

Expanding the product on the left-hand side we find

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$$(e^{\theta_1} + 1) \cdots (e^{\theta_d} + 1) = 0.$$

Expanding the product on the left-hand side we find

$$\sum_{\alpha} e^{\alpha} = 0,$$

where the sum is over all $\alpha = \epsilon_1 \theta_1 + ... + \epsilon_d \theta_d$ with $\epsilon_1, ..., \epsilon_d \in \{0, 1\}$.

Transcendence of π

Proof (Cont).

Suppose precisely n of α's are nonzero, say $\alpha_1, ..., \alpha_n$. Then

$$q + e^{\alpha_1} + ... + e^{\alpha_n} = 0, \tag{2}$$

where $q = 2^d - n$. Let c be a positive integer such that $c\theta_1, ..., c\theta_d$ are algebraic integers.

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$$f(x) := (cx)^{p-1}(cx - c\alpha_1)^p \cdots (cx - c\alpha_n)^p,$$

where p is a prime. Then $f \in \overline{\mathbb{Z}}[x]$ with degree $m = (n+1)p - 1$.

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$$J := \sum_{k=1}^{n} I(\alpha_k).$$

Proof (Cont).

By Lemma 2 and (2) we have

$$J = -q\sum_{j=0}^{m} f^{(j)}(0) - \sum_{j=0}^{m}\sum_{k=1}^{n} f^{(j)}(\alpha_k).$$

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$$g(x) := f(x/c) = x^{p-1}(x - c\alpha_1)^p \cdots (x - c\alpha_n)^p.$$

Then

$$(x - c\alpha_1) \cdots (x - c\alpha_n) \in \mathbb{Z}[x]$$

and thus $f, g \in \mathbb{Z}[x]$. By Lemma 3 we have

$$\frac{1}{j!} \sum_{k=1}^{n} g^{(j)}(c\alpha_k) \in \mathbb{Z}.$$

Proof (Cont).

Since $f(x) = g(cx)$, we have

$$\frac{1}{j!}\sum_{k=1}^{n} f^{(j)}(\alpha_k) = \frac{c^j}{j!}\sum_{k=1}^{n} g^{(j)}(c\alpha_k) \in \mathbb{Z}.$$

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Observe that $f^{(j)}(\alpha_k) = 0$ if $0 \leq j < p-1$ and

$$\sum_{k=1}^{n} f^{(j)}(\alpha_k) \equiv 0 \pmod{j!}$$

if $j \geq p$. So we have

$$\sum_{j=0}^{m}\sum_{k=1}^{n} f^{(j)}(\alpha_k) \equiv 0 \pmod{p!}.$$

Transcendence of π

Proof (Cont).

By Lemma 1, we have $f^{(j)}(0) \equiv 0 \pmod{j!}$ for all $j \geq p$. Note also that $f^{(j)}(0) = 0$ if $0 \leq j < p-1$ and that

$$f^{(p-1)}(0) = c^{p-1}(-1)^{np}(p-1)![(c\alpha_1)\cdots(c\alpha_n)]^p.$$

Hence, we have

$$J \equiv c^{p-1}(-1)^{np}(p-1)![(c\alpha_1)\cdots(c\alpha_n)]^p \pmod{p!}.$$

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Choose p sufficiently large so that $p > \max(c, |(c\alpha_1)\cdots(c\alpha_n)|)$. Then $(p-1)! \mid J$ but $p! \nmid J$. This implies that $|J| \geq (p-1)!$.

Proof (Cont).

On the other hand, we have

$$\bar{f}(|\alpha_k|) \le (cA)^m 2^{np}$$

for all $1 \le k \le n$, where $A = \max_{1 \le k \le n} |\alpha_k|$.

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for all $1 \leq k \leq n$, where $A = \max_{1 \leq k \leq n} |\alpha_k|$. It follows from Lemma 2 that

$$|J| \leq \sum_{k=1}^{n} |I(\alpha_k)| \leq \sum_{k=1}^{n} k e^k \bar{f}(|\alpha_k|) \ll B^p$$

for some constant $B > 0$. So $(p-1)! \ll B^p$. This is impossible when p is sufficiently large. $\square$

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The Lindemann-Weierstrass Theorem

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Theorem (Lindemann-Weierstrass, 1885)

Let $\alpha_1, \ldots, \alpha_n$ be distinct algebraic numbers. Then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are linearly independent over $\overline{\mathbb{Q}}$.

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Corollary 2

Let $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$. Then $\log\alpha$ is transcendental.

The Gelfond–Schneider Theorem

In 1900 Hilbert raised the following question at the International Congress of Mathematicians held in Paris.

Hilbert's 7$^{\text{th}}$ Problem

Is α^β transcendental for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and any irrational $\beta \in \overline{\mathbb{Q}}$?

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This problem was solved by Gelfond and Schneider independently in 1934.

Theorem (Gelfond-Schneider, 1934)

α^β is transcendental for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and any irrational $\beta \in \overline{\mathbb{Q}}$.

The Gelfond–Schneider Theorem

In 1900 Hilbert raised the following question at the International Congress of Mathematicians held in Paris.

Hilbert's 7^{th} Problem

Is α^β transcendental for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and any irrational $\beta \in \overline{\mathbb{Q}}$?

This problem was solved by Gelfond and Schneider independently in 1934.

Theorem (Gelfond-Schneider, 1934)

α^β is transcendental for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ and any irrational $\beta \in \overline{\mathbb{Q}}$.

Corollary 3

$2^{\sqrt{2}}$ and e^π are both transcendental.

Baker's Theorem

The Gelfond–Schneider theorem is equivalent to the statement that if $\alpha, \beta \in \overline{\mathbb{Q}} \setminus \{0\}$ are such that $\log \alpha, \log \beta$ are linearly independent over $\mathbb{Q}$, then they are linearly independent over $\overline{\mathbb{Q}}$.

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In 1966 Baker obtained the following generalization.

Theorem (Baker, 1966)

If $\alpha_1, \ldots, \alpha_n$ are nonzero algebraic numbers such that $\log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\mathbb{Q}$, then $1, \log \alpha_1, \ldots, \log \alpha_n$ are linearly independent over $\overline{\mathbb{Q}}$.

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Corollary 4

$e^{\beta_0} \alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n}$ is transcendental for any $\alpha_1, \dots, \alpha_n, \beta_0, \dots, \beta_n \in \overline{\mathbb{Q}} \setminus \{0\}$.

Baker's Theorem and Thue's Equation

Baker obtained an effective lower bound for the absolute value of

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + ... + \beta_n \log \alpha_n$$

in terms of the degrees and heights of $\alpha_1, ..., \alpha_n \in \overline{\mathbb{Q}} \setminus \{0\}$, $\beta_0, ..., \beta_n \in \overline{\mathbb{Q}}$: if $\Lambda \neq 0$, then $|\Lambda| > B^{-C}$, where $B, C > 0$ are effective constants.

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Theorem (Thue, 1909)

Let

$$f(x, y) := \sum_{k=0}^{n} a_k x^k y^{n-k} \in \mathbb{Z}[x]$$

be an irreducible polynomial of degree $n \geq 3$. Then for every $m \in \mathbb{Z} \setminus \{0\}$, the equation $f(x, y) = m$ has finitely many solutions $(x, y) \in \mathbb{Z}^2$.

Table of Contents

Open Problems

- Are $e \pm \pi, e\pi, \pi^e$ irrational/transcendental?
- Is Euler's constant

$$\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log N \right)$$

  irrational/transcendental?
- In 1978 Apéry established the irrationality of $\zeta(3)$, where

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

  Is $\zeta(3)$ transcendental? What about $\zeta(k)$ for odd $k \geq 5$?
- The Gelfond-Schneider theorem implies that $\log 2, \log 3$ are linearly independent over $\overline{\mathbb{Q}}$. Are they algebraically independent?

Open Problems

- The six exponentials theorem, proved independently by Lang and Ramachandra, states that if $x_1, x_2, x_3 \in \mathbb{C}$ and $y_1, y_2 \in \mathbb{C}$ are such that x_1, x_2, x_3 are linearly independent over $\mathbb{Q}$ and y_1, y_2 are linearly independent over $\mathbb{Q}$, then at least one of the following six numbers

$$e^{x_i y_j}, \quad 1 \le i \le 3,\ 1 \le j \le 2,$$

is transcendental. It is conjectured that if $x_1, x_2 \in \mathbb{C}$ and $y_1, y_2 \in \mathbb{C}$ are such that each pair is linearly independent over $\mathbb{Q}$, then at least one of the four numbers $e^{x_i y_j}$ is transcendental. This is now referred to as the four exponentials conjecture.

Thank you for your attention!